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Implementation of the Bäcklund transformations for the Ablowitz–Ladik hierarchy

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Abstract

The derivation of the Bäcklund transformations (BTs) is a standard problem of the theory of the integrable systems. Here, I discuss the equations describing the BTs for the Ablowitz–Ladik hierarchy (ALH), which have been already obtained by several authors. The main aim of this work is to solve these equations. This can be done in the framework of the so-called functional representation of the ALH, when an infinite number of the evolutionary equations are replaced, using the Miwa's shifts, with a few equations linking tau-functions with different arguments. It is shown that starting from these equations it is possible to obtain explicit solutions of the BT equations. In other words, the main result of this work is a presentation of the discrete BTs as a superposition of an infinite number of evolutionary flows of the hierarchy. These results are used to derive the superposition formulae for the BTs as well as pure soliton solutions.

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1. Introduction

The present paper is devoted to the Bäcklund transformations (BTs) for the Ablowitz–Ladik hierarchy (ALH) [1, 2]. Since the discovery of the inverse scattering transform (IST) these transformations have become one of the most powerful tools for generating solutions for nonlinear integrable equations (one can find the discussion of various aspects of this subject in the monograph [3]). The scheme of the solution generation can be briefly described as follows. One of the main points of the IST is the so-called zero-curvature representation (ZCR), when a nonlinear equation is presented as a compatibility condition for some overdetermined auxiliary *linear* system. To this linear system one can apply the Darboux transform. This leads to some new solutions of the auxiliary problem and, consequently, of the initial nonlinear equation,

which are determined (implicitly in general case) by the solutions we started with. In other words, the Darboux transformations for the auxiliary linear equations give us the algorithm to construct the Bäcklund transformations for the nonlinear one. This scheme was applied in 1970s to almost all known integrable nonlinear equations (one can find the main results of these studies in the book [4] and references therein). This construction has been implemented by several authors for the ALH [5–8] who obtained the systems describing the corresponding BTs.

The main aim of this work is not to derive these equations but to try to *solve* them. To do this I will combine the standard algorithm based on the ZCR with another approach to the BTs, which is associated mostly with the KP or 2D Toda lattice equations and which is based on rewriting the equation in question in terms of the so-called Miwa variables (see below). One can find a rather detailed description of this technique in the paper by Adler and van Moerbeke [9]. During a long time the IST and the Miwa's representation were used, in some sense, independently: some results have been derived using one approach, some using the other. One of the motives to write this work is to demonstrate that the combination of these two methods is a very useful strategy which gives possibility of solving some problems which are difficult to solve using only one of them. The BTs for the ALH discussed in this paper is a bright example of such problem. It turns out that combining both these approaches is rather fruitful and gives possibility of obtaining formally *explicit* expressions for the BTs which is the main result of this paper.

The plan of the paper is as follows. After deriving the equations describing the BTs (section 2) and presenting the necessary facts related to the functional representation of the ALH (section 3) I will obtain the formal solution of the Bäcklund equations, i.e. present the explicit realization of the BTs for the ALH (section 4). These will be used to calculate the superposition of several BTs (section 5) and then to rederive the N -soliton solutions (section 6).

2. Derivation of the Bäcklund equations

All equations of the ALH, which is an infinite set of ordinal differential-difference equations, can be presented as the compatibility condition for the linear system

$$\Psi_{n+1} = U_n \Psi_n \quad (2.1)$$

$$\partial_j \Psi_n = V_n^{(j)} \Psi_n \quad (2.2)$$

where ∂_j stands for $\partial/\partial z_j$, z_j s are an infinite set of variables (times), Ψ_n is a two column (or 2×2 matrix), U_n is given by

$$U_n = U_n(\lambda) = \begin{pmatrix} \lambda & r_n \\ q_n & \lambda^{-1} \end{pmatrix} \quad (2.3)$$

and $V_n^{(j)}$ s are some 2×2 matrices whose elements are some polynomials in λ , λ^{-1} and depend on $q_n, r_n, q_{n\pm 1}, r_{n\pm 1}, \dots$. In what follows, we will not use their explicit form (one can find how to construct these matrices in the pioneering work [1] or, e.g., in the book [10]). To provide the self-consistency of the system (2.1), (2.2), the matrices V_n have to satisfy the following equations:

$$\partial_j U_n = V_{n+1}^{(j)} U_n - U_n V_n^{(j)} \quad (2.4)$$

which, when rewritten in terms of q_n s and r_n s, constitute the ALH.

From the viewpoint of the IST, the BTs are the transformations of the form

$$\tilde{\Psi}_n = M_n \Psi_n \quad (2.5)$$

which are compatible with the $n \rightarrow n+1$ shift. This means that $\tilde{\Psi}_n$ should satisfy the equation similar to (2.1) with the matrix U_n being replaced with some other matrix \tilde{U}_n of the same structure, which leads to the following equation for the matrix M_n :

$$M_{n+1}U_n - \tilde{U}_n M_n = 0 \quad (2.6)$$

where

$$U_n = \begin{pmatrix} \lambda & r_n \\ q_n & \lambda^{-1} \end{pmatrix}, \quad \tilde{U}_n = \begin{pmatrix} \lambda & \tilde{r}_n \\ \tilde{q}_n & \lambda^{-1} \end{pmatrix}. \quad (2.7)$$

Rewriting this matrix equation as a scalar system for the elements of M_n ,

$$M_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad (2.8)$$

one comes to the equations

$$\lambda a_{n+1} + q_n b_{n+1} = \lambda a_n + \tilde{r}_n c_n \quad (2.9)$$

$$\lambda^{-1} d_{n+1} + r_n c_{n+1} = \lambda^{-1} d_n + \tilde{q}_n b_n \quad (2.10)$$

$$\lambda^{-1} b_{n+1} + r_n a_{n+1} = \lambda b_n + \tilde{r}_n d_n \quad (2.11)$$

$$\lambda c_{n+1} + q_n d_{n+1} = \lambda^{-1} c_n + \tilde{q}_n a_n. \quad (2.12)$$

A simple analysis of (2.9)–(2.12) leads to the conclusion that there exist two types of solutions of this system (compare, e.g., with the transformations of the first and second kinds in the paper [7]).

2.1. \mathbb{B} -transformations

This type of transformations is determined by the following choice of the dependence of M_n on λ :

$$M_n = \begin{pmatrix} \lambda^2 a + a' \alpha_n & \lambda a r_n \\ \lambda a \tilde{q}_{n-1} & d \end{pmatrix}. \quad (2.13)$$

System (2.9)–(2.12) now becomes

$$a'(\alpha_{n+1} - \alpha_n) + a(q_n r_{n+1} - \tilde{q}_{n-1} \tilde{r}_n) = 0 \quad (2.14)$$

$$a r_{n+1} + a' r_n \alpha_{n+1} - d \tilde{r}_n = 0 \quad (2.15)$$

$$a \tilde{q}_{n-1} + a' \tilde{q}_n \alpha_n - d q_n = 0. \quad (2.16)$$

By introducing the tau-functions of the ALH,

$$q_n = \frac{\sigma_n}{\tau_n}, \quad r_n = \frac{\rho_n}{\tau_n}, \quad \tau_n^2 = \tau_{n-1} \tau_{n+1} + \rho_n \sigma_n, \quad (2.17)$$

one can find α_n ,

$$\alpha_n = \frac{\tau_{n-1} \tilde{\tau}_n}{\tau_n \tilde{\tau}_{n-1}}, \quad (2.18)$$

while the remaining equations can be rewritten as

$$a\rho_n\tilde{\tau}_{n-1} + a'\rho_{n-1}\tilde{\tau}_n - d\tau_n\tilde{\rho}_{n-1} = 0 \quad (2.19)$$

$$a\tau_n\tilde{\sigma}_{n-1} + a'\tau_{n-1}\tilde{\sigma}_n - d\sigma_n\tilde{\tau}_{n-1} = 0. \quad (2.20)$$

These equations are enough to meet (2.6). However, the condition $(\tilde{\tau}_n)^2 = \tilde{\tau}_{n-1}\tilde{\tau}_{n+1} + \tilde{\rho}_n\tilde{\sigma}_n$ imposes some additional restrictions on the transformed (tilded) tau-functions, which can be presented in the form

$$a'\rho_n\tilde{\sigma}_n - d'\tau_n\tilde{\tau}_n + d\tau_{n+1}\tilde{\tau}_{n-1} = 0. \quad (2.21)$$

The system (2.19)–(2.21) completely determines this kind of transformations, which in what follows I will denote by the symbol \mathbb{B} instead of tilde, $\mathbb{B}f = \tilde{f}$.

2.2. $\overline{\mathbb{B}}$ -transformations

This type of transformations corresponds to the following choice of the M -matrix:

$$\overline{M}_n(\lambda) = \begin{pmatrix} \bar{a} & \lambda^{-1}\bar{d}\tilde{r}_{n-1} \\ \lambda^{-1}\bar{d}q_n & \bar{d}'\bar{\alpha}_n + \lambda^{-2}\bar{d} \end{pmatrix}. \quad (2.22)$$

In what follows, all quantities related to this kind of transformations, which I will denote by the symbol $\overline{\mathbb{B}}$, will be marked with overbars, which *does not mean complex conjugation*. After calculating $\bar{\alpha}_n$, $\bar{\alpha}_n = \tau_{n-1}\tilde{\tau}_n/\tau_n\tilde{\tau}_{n-1}$, one can derive equations similar to (2.19)–(2.21),

$$\bar{d}\tau_n\tilde{\rho}_{n-1} + \bar{d}'\tau_{n-1}\tilde{\rho}_n - \bar{a}\rho_n\tilde{\tau}_{n-1} = 0 \quad (2.23)$$

$$\bar{d}\sigma_{n+1}\tilde{\tau}_n + \bar{d}'\sigma_n\tilde{\tau}_{n+1} - \bar{a}\tau_{n+1}\tilde{\sigma}_n = 0 \quad (2.24)$$

$$\bar{a}\tau_{n+1}\tilde{\tau}_{n-1} - \bar{a}'\tau_n\tilde{\tau}_n + \bar{d}'\sigma_n\tilde{\rho}_n = 0. \quad (2.25)$$

Thus, we have obtained equations describing two types of elementary Bäcklund transformations:

$$\mathbb{B}: \begin{cases} a\rho_n(\mathbb{B}\tau_{n-1}) + a'\rho_{n-1}(\mathbb{B}\tau_n) - d\tau_n(\mathbb{B}\rho_{n-1}) = 0 \\ a\tau_n(\mathbb{B}\sigma_{n-1}) + a'\tau_{n-1}(\mathbb{B}\sigma_n) - d\sigma_n(\mathbb{B}\tau_{n-1}) = 0 \\ a'\rho_n(\mathbb{B}\sigma_n) - d'\tau_n(\mathbb{B}\tau_n) + d\tau_{n+1}(\mathbb{B}\tau_{n-1}) = 0 \end{cases} \quad (2.26)$$

and

$$\overline{\mathbb{B}}: \begin{cases} \bar{d}\tau_{n+1}(\overline{\mathbb{B}}\rho_n) + \bar{d}'\tau_n(\overline{\mathbb{B}}\rho_{n+1}) - \bar{a}\rho_{n+1}(\overline{\mathbb{B}}\tau_n) = 0 \\ \bar{d}\sigma_{n+1}(\overline{\mathbb{B}}\tau_n) + \bar{d}'\sigma_n(\overline{\mathbb{B}}\tau_{n+1}) - \bar{a}\tau_{n+1}(\overline{\mathbb{B}}\sigma_n) = 0 \\ \bar{a}\tau_{n+1}(\overline{\mathbb{B}}\tau_{n-1}) - \bar{a}'\tau_n(\overline{\mathbb{B}}\tau_n) + \bar{d}'\sigma_n(\overline{\mathbb{B}}\rho_n) = 0. \end{cases} \quad (2.27)$$

Writing equations (2.26) and (2.27) is a crucial step in constructing the BTs for the equations in question. Now one can, in principle, forget about the ZCR for the ALH which was our starting point and consider the bilinear systems (2.26), (2.27) as the definitions of our transformations (which could be derived by other methods, say, the Hirota's bilinear approach).

Usually from the practical viewpoint the BTs are used as follows. One takes simple solutions of the integrable system in question (the ALH in our case) and then tries to solve the Bäcklund equations (equations (2.26) or (2.27) in our case) to obtain some more complicated ones (probably the first who applied BTs for the generation of solutions of soliton equations were Seeger *et al* [11]). It is possible to solve the Bäcklund equations explicitly if we start with some trivial solutions (e.g., constant or pure soliton), but to do it in general case is a task not less difficult than to solve the initial one. Nevertheless, it turns out (and it is the main point of

this work) that it is possible to derive some (maybe formal) solutions for the Bäcklund system (equations (2.26) or (2.27)). This is difficult to do if one restricts oneself to some particular integrable equation (say, to the discrete nonlinear Schrödinger equation or discrete modified KdV equation, both of which belong to the ALH). At the same time, information contained in *all* equations of the hierarchy is sufficient to tackle this problem. We can construct a discrete (Bäcklund) flow of an infinite number of continuous ones. In other words, if we know that our tau-functions are solutions of the ALH (i.e., of the infinite set of compatible differential equations), then we can use them to derive solutions of (2.26) and (2.27). This will be done in section 4 after presenting some basic facts about the ALH in the next section.

3. Ablowitz–Ladik hierarchy

For our purposes we need the so-called functional representation of the ALH, which has been elaborated in [12, 13], when the infinite set of differential-difference equations of the ALH is presented as a few functional equations relating tau-functions of different (shifted) arguments. This way of rewriting a hierarchy using the Miwa's shifts (in the form of the so-called Fay's identities) is very convenient when one studies the hierarchy 'as a whole' especially in the case like ours when we want to deal with *all* evolutionary flows simultaneously. The reader can find the derivation of this form of the ALH together with some formulae which will be used below in [13]. However, to simplify the following calculations it seems fruitful to rewrite the results of [13] in a more general way. To this end it should be noted that some formulae of the previous section, as well as of [13], are related by the shifts $\rho_n \rightarrow \tau_n \rightarrow \sigma_n$. That is why I rewrite the tau-functions as

$$\rho_n = \tau_n^{-1}, \quad \tau_n = \tau_n^0, \quad \sigma_n = \tau_n^1 \quad (3.1)$$

and introduce a double infinite set of tau-functions τ_n^m , $m = \pm 2, \pm 3, \dots$, using a generalization of the relation written in (2.17),

$$(\tau_n^m)^2 = \tau_{n-1}^m \tau_{n+1}^m + \tau_n^{m-1} \tau_n^{m+1}. \quad (3.2)$$

Thus, I define

$$\tau_n^m = \frac{1}{\tau_n^{m-2}} [(\tau_n^{m-1})^2 - \tau_{n-1}^{m-1} \tau_{n+1}^{m-1}] \quad \text{for } m = 2, 3, \dots \quad (3.3)$$

and

$$\tau_n^m = \frac{1}{\tau_n^{m+2}} [(\tau_n^{m+1})^2 - \tau_{n-1}^{m+1} \tau_{n+1}^{m+1}] \quad \text{for } m = -2, -3, \dots \quad (3.4)$$

It can be shown that all equations of the ALH are compatible with (3.3) and (3.4) and all equations of the extended version of the ALH can be obtained from the following ones:

$$0 = \zeta \tau_{n-1}^m (\mathbb{E}_\zeta \tau_{n+1}^{m+1}) + \tau_n^{m+1} (\mathbb{E}_\zeta \tau_n^m) - \tau_n^m (\mathbb{E}_\zeta \tau_n^{m+1}) \quad (3.5)$$

$$0 = \tau_{n-1}^m (\mathbb{E}_\zeta \tau_{n+1}^m) + \tau_n^{m-1} (\mathbb{E}_\zeta \tau_n^{m+1}) - \tau_n^m (\mathbb{E}_\zeta \tau_n^m) \quad (3.6)$$

(the positive subhierarchy) and

$$0 = \zeta \tau_{n+1}^m (\overline{\mathbb{E}}_\zeta \tau_{n-1}^{m+1}) + \tau_n^{m+1} (\overline{\mathbb{E}}_\zeta \tau_n^m) - \tau_n^m (\overline{\mathbb{E}}_\zeta \tau_n^{m+1}) \quad (3.7)$$

$$0 = \tau_{n+1}^m (\overline{\mathbb{E}}_\zeta \tau_{n-1}^m) + \tau_n^{m-1} (\overline{\mathbb{E}}_\zeta \tau_n^{m+1}) - \tau_n^m (\overline{\mathbb{E}}_\zeta \tau_n^m) \quad (3.8)$$

(the negative one). Here, the symbols \mathbb{E}_ξ and $\overline{\mathbb{E}}_\eta$ stand for the Miwa’s shifts

$$\mathbb{E}_\zeta f(z, \bar{z}) = f(z + i[\zeta], \bar{z}), \quad \overline{\mathbb{E}}_\zeta f(z, \bar{z}) = f(z, \bar{z} + i[\zeta]) \tag{3.9}$$

which are defined for functions of an infinite number of variables,

$$f(z, \bar{z}) = f(z_1, z_2, z_3, \dots, \bar{z}_1, \bar{z}_2, \bar{z}_3, \dots), \tag{3.10}$$

by

$$f(z + \alpha[\xi], \bar{z} + \beta[\eta]) = f(z_1 + \alpha\xi, z_2 + \alpha\xi^2/2, z_3 + \alpha\xi^3/3, \dots, \bar{z}_1 + \beta\eta, \bar{z}_2 + \beta\eta^2/2, \bar{z}_3 + \beta\eta^3/3, \dots). \tag{3.11}$$

The extended version of the ALH (3.5)–(3.8) was written in the paper [14] by Sadakane who showed, using the free fermions approach, that the ALH arises from a reduction of the two-component Toda lattice hierarchy, discussed the ALH in the context of the affine Lie algebra that acts on the universal Grassmann manifold and clarified the universality of the ALH.

In what follows, we also need some formulae from [13] describing superposition of Miwa’s shifts. They are collected in their extended form (i.e., in terms of the tau-functions of the extended ALH hierarchy, τ_n^m) in the appendix.

Now we have all necessary to ‘solve’ equations (2.26) and (2.27) which can be rewritten as

$$\mathbb{B}: \begin{cases} a\tau_{n+1}^m(\mathbb{B}\tau_n^{m+1}) + a'\tau_n^m(\mathbb{B}\tau_{n+1}^{m+1}) - d\tau_{n+1}^{m+1}(\mathbb{B}\tau_n^m) = 0 \\ a'\tau_n^{m-1}(\mathbb{B}\tau_n^{m+1}) - d'\tau_n^m(\mathbb{B}\tau_n^m) + d\tau_{n+1}^m(\mathbb{B}\tau_{n-1}^m) = 0 \end{cases} \tag{3.12}$$

and

$$\overline{\mathbb{B}}: \begin{cases} \bar{d}\tau_{n+1}^{m+1}(\overline{\mathbb{B}}\tau_n^m) + \bar{d}'\tau_n^{m+1}(\overline{\mathbb{B}}\tau_{n+1}^m) - \bar{a}\tau_{n+1}^m(\overline{\mathbb{B}}\tau_n^{m+1}) = 0 \\ \bar{a}\tau_{n+1}^m(\overline{\mathbb{B}}\tau_{n-1}^m) - \bar{a}'\tau_n^m(\overline{\mathbb{B}}\tau_n^m) + \bar{d}'\tau_n^{m+1}(\overline{\mathbb{B}}\tau_n^{m-1}) = 0. \end{cases} \tag{3.13}$$

4. Solution of the Bäcklund equations

As it was mentioned at the end of section 2, one can obtain, starting from the fact that τ_n^m solve the ALH equations, some particular solutions of the Bäcklund equations (3.12) and (3.13). Indeed, by multiplying (3.5) by $(d/a)^m(d/d')^n$ one can come to the conclusion that the quantity

$$\mathbb{B}^{(1)}\tau_n^m = \left(\frac{d}{a}\right)^m \left(\frac{d}{d'}\right)^n \mathbb{E}_\zeta \tau_{n+1}^m \tag{4.1}$$

solves the first of equations (3.12) provided

$$\zeta = -\frac{a'd}{ad'}. \tag{4.2}$$

In a similar way, one can check that (4.1) also satisfies the second one. Thus, we have derived some BT which can be expressed in terms of the Miwa’s shifts (i.e., the evolutionary flows) accompanied by the shifts of the site index n . Of course, the transform $\mathbb{B}^{(1)}$ is almost trivial and is not interesting from the viewpoint of applications. However, this transformation is not the only one and it is possible to derive three other similar transformations, involving both positive and negative shifts:

$$\mathbb{B}^{(1)}\tau_n^m = \left(\frac{d}{a}\right)^m \left(\frac{d}{d'}\right)^n \mathbb{E}_\zeta \tau_{n+1}^m \tag{4.3}$$

$$\mathbb{B}^{(2)} \tau_n^m = \left(\frac{d'}{a'}\right)^m \left(-\frac{a}{a'}\right)^n \mathbb{E}_\zeta^{-1} \tau_n^{m-1} \tag{4.4}$$

$$\mathbb{B}^{(3)} \tau_n^m = \left(\frac{d'}{a'}\right)^m \left(\frac{d}{d'}\right)^n \overline{\mathbb{E}}_{1/\zeta} \tau_n^m \tag{4.5}$$

$$\mathbb{B}^{(4)} \tau_n^m = \left(\frac{d}{a}\right)^m \left(-\frac{a}{a'}\right)^n \overline{\mathbb{E}}_{1/\zeta}^{-1} \tau_{n+1}^{m-1} \tag{4.6}$$

with

$$\zeta = -\frac{a'd}{ad'} \tag{4.7}$$

which can be used to construct more rich ones,

$$\mathbb{B} \tau_n^m = \sum_{k=1}^4 u^{(k)} \mathbb{B}^{(k)} \tau_n^m. \tag{4.8}$$

It is an important moment. Each of $\mathbb{B}^{(k)} \tau_n^m$ solves the *bilinear* ALH equations (3.5)–(3.8). However their linear combination does not satisfy then automatically. Nevertheless, the integrable bilinear equations possess some peculiar features and it turns out that by imposing some restrictions on the functions $u^{(k)}$ in (4.8) one can make $\mathbb{B} \tau_n^m$ to solve the ALH equations. To do this, one needs to study the properties of the products of the Miwa’s shifts and the trivial BTs $\mathbb{B}^{(k)}$.

From the superposition formulae for the Miwa’s shifts presented in the appendix, one can deduce that all $\mathbb{B}^{(k)}$ satisfy the similar identities:

$$\lambda^{(k)}(\xi) \tau_n^m (\mathbb{E}_\xi \mathbb{B}^{(k)} \tau_n^m) = (\mathbb{E}_\xi \tau_n^m) (\mathbb{B}^{(k)} \tau_n^m) + \xi \frac{a}{a'} (\mathbb{E}_\xi \tau_{n+1}^m) (\mathbb{B}^{(k)} \tau_{n-1}^m) \tag{4.9}$$

where

$$\lambda^{(1)}(\xi) = a(\zeta, \xi)/\zeta \tag{4.10}$$

$$\lambda^{(2)}(\xi) = c(\xi, \zeta) \tag{4.11}$$

$$\lambda^{(3)}(\xi) = e(\xi, 1/\zeta) \tag{4.12}$$

$$\lambda^{(4)}(\xi) = g(\xi, 1/\zeta). \tag{4.13}$$

Thus, if one takes the functions $u^{(k)}$ depending on the variables z_j according to the rule

$$\frac{\mathbb{E}_\xi u^{(k)}}{u^{(k)}} = \frac{\lambda^{(k)}(\xi)}{\lambda(\xi)} \tag{4.14}$$

where $\lambda(\xi)$ is an arbitrary function satisfying the restriction $\lambda(0) = 1$, then $\mathbb{B} \tau_n^m$ satisfies

$$\lambda(\xi) \mathbb{E}_\xi (\mathbb{B} \tau_n^m) = \frac{1}{\tau_n^m} \left[(\mathbb{E}_\xi \tau_n^m) (\mathbb{B} \tau_n^m) + \xi \frac{a}{a'} (\mathbb{E}_\xi \tau_{n+1}^m) (\mathbb{B} \tau_{n-1}^m) \right]. \tag{4.15}$$

Now by straightforward algebra one can show that the transformation

$$\tilde{\tau}_n^m \rightarrow \frac{1}{\lambda(\xi) \tau_n^m} \left[(\mathbb{E}_\xi \tau_n^m) \tilde{\tau}_n^m + \xi \frac{a}{a'} (\mathbb{E}_\xi \tau_{n+1}^m) \tilde{\tau}_{n-1}^m \right] \tag{4.16}$$

is compatible with the evolutionary equations: if one substitutes into, e.g., (3.5) τ_n^m with $\tilde{\tau}_n^m$ (which is a solution of the Bäcklund equations (3.12)) and $\mathbb{E}_\xi \tau_n^m$ with the rhs of (4.16) then the resulting expression will be identically zero.

Analogously, $\mathbb{B}\tau_n^m$ solve equations of the negative subhierarchy, if the dependence of the functions u_k on the variables \bar{z}_j is given by

$$\frac{\bar{\mathbb{E}}_\eta u^{(k)}}{u^{(k)}} = \frac{\mu^{(k)}(\eta)}{\mu(\eta)} \quad (4.17)$$

where

$$\mu^{(1)}(\eta) = e(\zeta, \eta) \quad (4.18)$$

$$\mu^{(2)}(\eta) = g(\zeta, \eta) \quad (4.19)$$

$$\mu^{(3)}(\eta) = \zeta \bar{a}(1/\zeta, \eta) \quad (4.20)$$

$$\mu^{(4)}(\eta) = \bar{c}(1/\zeta, \eta) \quad (4.21)$$

and $\mu(\eta)$ is an arbitrary function with $\mu(0) = 1$.

Equations (4.14) and (4.17) can be easily solved by the ansatz

$$u^{(k)}(z, \bar{z}) = \exp \left\{ \sum_{j=1}^{\infty} \omega_{kj} z_j + \bar{\omega}_{kj} \bar{z}_j \right\} \quad (4.22)$$

where

$$\sum_{j=1}^{\infty} \omega_{kj} \frac{\xi^j}{j} = -i \ln \frac{\lambda^{(k)}(\xi)}{\lambda(\xi)} \quad (4.23)$$

and

$$\sum_{j=1}^{\infty} \bar{\omega}_{kj} \frac{\eta^j}{j} = -i \ln \frac{\mu^{(k)}(\eta)}{\mu(\eta)}, \quad (4.24)$$

i.e., ω_{kj} and $\bar{\omega}_{kj}$ are the Taylor coefficients for the logarithmic derivatives of the functions $\lambda^{(k)}/\lambda$ and $\mu^{(k)}/\mu$:

$$\sum_{j=1}^{\infty} \omega_{kj} \xi^{j-1} = -i \frac{d}{d\xi} \ln \frac{\lambda^{(k)}(\xi)}{\lambda(\xi)} \quad (4.25)$$

$$\sum_{j=1}^{\infty} \bar{\omega}_{kj} \eta^{j-1} = -i \frac{d}{d\eta} \ln \frac{\mu^{(k)}(\eta)}{\mu(\eta)}. \quad (4.26)$$

These formulae are nothing but the dispersion laws for the given boundary conditions (which determine the quantities $a(\xi, \eta)$, $e(\xi, \eta)$, etc, and hence all the functions $\lambda^{(k)}(\xi)$ and $\mu^{(k)}(\eta)$).

Similar algorithm can be developed for the $\bar{\mathbb{B}}$ transformations. Starting from almost trivial ones,

$$\bar{\mathbb{B}}^{(1)} \tau_n^m = \left(\frac{\bar{d}}{\bar{a}} \right)^m \left(\frac{\bar{a}}{\bar{a}'} \right)^n \bar{\mathbb{E}}_{\bar{\zeta}}^{-1} \tau_{n+1}^m \quad (4.27)$$

$$\bar{\mathbb{B}}^{(2)} \tau_n^m = \left(\frac{\bar{d}'}{\bar{a}'} \right)^m \left(-\frac{\bar{d}}{\bar{d}'} \right)^n \bar{\mathbb{E}}_{\bar{\zeta}} \tau_n^{m+1} \quad (4.28)$$

$$\bar{\mathbb{B}}^{(3)} \tau_n^m = \left(\frac{\bar{d}'}{\bar{a}'} \right)^m \left(\frac{\bar{a}}{\bar{a}'} \right)^n \bar{\mathbb{E}}_{1/\bar{\zeta}}^{-1} \tau_n^m \quad (4.29)$$

$$\overline{\mathbb{B}}^{(4)} \tau_n^m = \left(\frac{\bar{d}}{\bar{a}}\right)^m \left(-\frac{\bar{d}}{\bar{d}'}\right)^n \mathbb{E}_{1/\bar{\zeta}} \tau_{n+1}^{m+1} \tag{4.30}$$

with

$$\bar{\zeta} = -\frac{\bar{a}\bar{d}'}{\bar{a}'\bar{d}}, \tag{4.31}$$

one can construct the more general transformations

$$\overline{\mathbb{B}} \tau_n^m = \sum_{k=1}^4 \bar{u}^{(k)} \overline{\mathbb{B}}^{(k)} \tau_n^m \tag{4.32}$$

where the functions $\bar{u}^{(k)}$ should be determined from the equations

$$\frac{\mathbb{E}_{\xi} \bar{u}^{(k)}}{\bar{u}^{(k)}} = \frac{\bar{\lambda}^{(k)}(\xi)}{\bar{\lambda}(\xi)}, \quad \frac{\overline{\mathbb{E}}_{\eta} \bar{u}^{(k)}}{\bar{u}^{(k)}} = \frac{\bar{\mu}^{(k)}(\eta)}{\bar{\mu}(\eta)} \tag{4.33}$$

with

$$\begin{aligned} \bar{\lambda}^{(1)}(\xi) &= g(\xi, \bar{\zeta}) & \bar{\mu}^{(1)}(\eta) &= \bar{c}(\eta, \bar{\zeta}) \\ \bar{\lambda}^{(2)}(\xi) &= e(\xi, \bar{\zeta}) & \bar{\mu}^{(2)}(\eta) &= \bar{a}(\bar{\zeta}, \eta)/\bar{\zeta} \\ \bar{\lambda}^{(3)}(\xi) &= c(\xi, 1/\bar{\zeta}) & \bar{\mu}^{(3)}(\eta) &= g(1/\bar{\zeta}, \eta) \\ \bar{\lambda}^{(4)}(\xi) &= \bar{\zeta} a(1/\bar{\zeta}, \xi) & \bar{\mu}^{(4)}(\eta) &= e(1/\bar{\zeta}, \eta) \end{aligned} \tag{4.34}$$

and arbitrary $\bar{\lambda}(\xi)$ and $\bar{\mu}(\eta)$ satisfying $\bar{\lambda}(0) = \bar{\mu}(0) = 1$.

So, we have derived the main result of this paper: we have *explicitly* constructed the BTs for the ALH. This was done by combining the elementary transformations $n \rightarrow n \pm 1, m \rightarrow m \pm 1$, the Miwa’s shifts and the multiplication by linear in n and m exponents. Each of these transformations is trivial and does not change the structure of solutions. However their combination can produce more rich BTs which give us possibility of obtaining more complicated solutions from simple ones. I will return to this point in the conclusion and discuss it using as an example the soliton-adding transformations of section 6. An important moment is that to obtain physically interesting transformations one has to use superposition of $\overline{\mathbb{B}}^{(k)}$ and $\overline{\mathbb{B}}^{(k)}$ with different ks . Moreover, one cannot restrict oneself with $k = 1, 2$ or $k = 3, 4$ only. Because of this it is impossible to obtain their infinitesimal versions by taking, say, the $\zeta \rightarrow 0$ and $\bar{\zeta} \rightarrow 0$ limits. That is an illustration of the discrete character of \mathbb{B} and $\overline{\mathbb{B}}$, contrary to the evolutionary operators \mathbb{E} and $\overline{\mathbb{E}}$.

5. Superposition of BTs

In the previous section we have derived the BTs, the transformations which commute with the $n \rightarrow n + 1$ shifts as well as with the evolutionary flows. Now our aim is to ensure the commutativity of these transformations. It turns out that this problem is, in some sense, more difficult than the previous ones. The commutativity with the $n \rightarrow n + 1$ shifts is due to the construction (2.6): all transformations (4.3)–(4.6) and (4.27)–(4.30) automatically satisfy (2.26) and (2.27) which were derived from (2.6). The commutativity with the evolutionary flows was achieved by the proper choice of the dependence of the functions $u^{(k)}$ and $\bar{u}^{(k)}$ ($k = 1, \dots, 4$) on z_j and \bar{z}_j ($j = 1, 2, \dots$) which is described by equations (4.14), (4.17) and (4.33).

Before proceeding further I have to introduce the following notation. Since now we will deal with different BTs, which will be distinguished by the additional subscript. The symbol

$\mathbb{B}_j^{(k)}$ means the k th elementary transform (one of (4.3)–(4.6)) constructed using the set of parameters a_j, a'_j, d_j and d'_j . Consequently, all the quantities ζ , the functions $\lambda^{(k)}, u^{(k)}, \lambda$ and others will also possess this index: $\zeta_j = -a'_j d_j / a_j d'_j, \lambda_j^{(k)}(\xi) = \lambda(\xi; a_j, a'_j, d_j, d'_j)$, etc. Finally \mathbb{B}_j is the BT corresponding to the superposition of $\mathbb{B}_j^{(k)}$ with coefficients $u_j^{(k)}$. The similar notation will be used for the BTs of the $\overline{\mathbb{B}}$ -type.

Now we can return to the superposition of BTs and I would like to start with an important remark. It should be noted that the commutativity of evolutionary flows and BTs has been done for \mathbb{B} and $\overline{\mathbb{B}}$ being taken as combinations of *four* elementary transformations, see (4.8) and (4.32). However, as will be shown below, if we want \mathbb{B} and $\overline{\mathbb{B}}$ to commute between themselves, we have to restrict ourselves to combinations of only *two* elementary BTs. Indeed, if we apply $\mathbb{B}_1^{(k)}$ to the \mathbb{B}_2 -transformed tau-function $\mathbb{B}_2 \tau_n^m$, we get

$$\beta_{12}^{(k)} \tau_n^{m-1} (\mathbb{B}_1^{(k)} \mathbb{B}_2 \tau_n^m) = \frac{d_2}{a_2} (\mathbb{B}_1^{(k)} \tau_n^m) (\mathbb{B}_2 \tau_n^{m-1}) - \frac{d_1}{a_1} (\mathbb{B}_1^{(k)} \tau_n^{m-1}) (\mathbb{B}_2 \tau_n^m) \quad (5.1)$$

where

$$\beta_{12}^{(1)} = \frac{a'_2}{a_2} \lambda_2(\zeta_1) \quad (5.2)$$

$$\beta_{12}^{(2)} = \frac{d_2}{a_2} \frac{1 - \zeta_1 / \zeta_2}{\lambda_2(\zeta_1)} \quad (5.3)$$

$$\beta_{12}^{(3)} = \zeta_1 \mu_2(1 / \zeta_1) \quad (5.4)$$

$$\beta_{12}^{(4)} = \frac{d'_2}{a_2} \frac{1 - \zeta_2 / \zeta_1}{\mu_2(1 / \zeta_1)}. \quad (5.5)$$

A simple analysis leads to the conclusion that it is impossible to choose the functions $\lambda_2(\xi)$ and $\mu_2(\eta)$ satisfying $\lambda_2(0) = \mu_2(0) = 1$ in a way to meet the condition

$$\beta_{12}^{(1)} = \beta_{12}^{(2)} = \beta_{12}^{(3)} = \beta_{12}^{(4)} \quad (5.6)$$

and to ensure the skew-symmetry of the left-hand side of (5.1) with respect to the interchange of the subscripts 1 and 2. However there is no problem, if one restricts oneself to only two elementary transformations. For example, if

$$\mathbb{B}_j = u_j^{(2)} \mathbb{B}_j^{(2)} + u_j^{(3)} \mathbb{B}_j^{(3)} \quad (5.7)$$

(this is the situation which corresponds to the bright soliton case, see also section 6), then the choice $\lambda_j(\xi) = 1, \mu_j(\eta) = 1 + \eta d_j / a_j$ together with the restriction $a'_j = d'_j$ leads to

$$\beta_{12} \tau_n^{m-1} (\mathbb{B}_{12} \tau_n^m) = \frac{d_2}{a_2} (\mathbb{B}_1 \tau_n^m) (\mathbb{B}_2 \tau_n^{m-1}) - \frac{d_1}{a_1} (\mathbb{B}_1 \tau_n^{m-1}) (\mathbb{B}_2 \tau_n^m) \quad (5.8)$$

with $\beta_{12} = d_2 / a_2 - d_1 / a_1$. Hereafter, the multi index of \mathbb{B} indicates the superposition of a few BTs

$$\mathbb{B}_{i_j \dots} \tau_n^m = \mathbb{B}_i \mathbb{B}_j \dots \tau_n^m \quad (5.9)$$

(after we have ensured the commutativity of different BTs, this notation has sense, because the result does not depend on the order of the application of BTs).

In a similar way, for

$$\mathbb{B}_j = u_j^{(1)} \mathbb{B}_j^{(1)} + u_j^{(3)} \mathbb{B}_j^{(3)} \quad (5.10)$$

(this is what one has in the dark soliton case), the choice $\lambda_j(\xi) = 1 + \xi a_j / a'_j, \mu_j(\eta) = 1 + \eta a'_j / a_j$, together with the restriction $d_j = d'_j$ leads again to (5.8), with $\beta_{12} = a'_2 / a_2 - a'_1 / a_1$.

To conclude this analysis, I would like to note that we are at the point where the boundary conditions play a crucial role. They entered earlier specifying the dispersion laws through the quantities $a(\xi, \eta)$, etc (see the previous section). But now their effect is a ‘qualitative’ one. They select the structure of the BTs and impose some restrictions on their parameters.

In the following consideration, I will not write explicitly the values of the coefficients like β_{jk} (though we already know that in the most relevant cases $\beta_{jk} = \zeta_j - \zeta_k$) because the ‘physical’ quantities, such as q_n, r_n of the ALH in the original setting, depend on ratios of the tau-functions. So, the constant (both with respect to the indices n, m and the times) multipliers (β_{jk} and similar ones which appear below) are not crucial.

By applying the Bäcklund equations (3.12) and (3.13) to the superposition formula (5.8) derived above, one can obtain a large number of the equivalent ones. Some of them are given by

$$\beta_{12} \tau_{n+1}^m (\mathbb{B}_{12} \tau_n^m) = \frac{a'_2}{a_2} (\mathbb{B}_1 \tau_n^m) (\mathbb{B}_2 \tau_{n+1}^m) - \frac{a'_1}{a_1} (\mathbb{B}_1 \tau_{n+1}^m) (\mathbb{B}_2 \tau_n^m) \tag{5.11}$$

$$\beta_{12} \tau_{n+1}^{m-1} (\mathbb{B}_{12} \tau_n^{m+1}) = \frac{d_1 d'_2}{a_1 a_2} (\mathbb{B}_1 \tau_n^m) (\mathbb{B}_2 \tau_{n+1}^m) - \frac{d'_1 d_2}{a_1 a_2} (\mathbb{B}_1 \tau_{n+1}^m) (\mathbb{B}_2 \tau_n^m). \tag{5.12}$$

In a similar way, one can derive formulae describing the product of the $\bar{\mathbb{B}}$ transformations

$$\bar{\beta}_{12} \tau_n^{m+1} (\bar{\mathbb{B}}_{12} \tau_n^m) = \frac{\bar{a}_2}{\bar{d}_2} (\bar{\mathbb{B}}_1 \tau_n^m) (\bar{\mathbb{B}}_2 \tau_n^{m+1}) - \frac{\bar{a}_1}{\bar{d}_1} (\bar{\mathbb{B}}_1 \tau_n^{m+1}) (\bar{\mathbb{B}}_2 \tau_n^m) \tag{5.13}$$

$$\bar{\beta}_{12} \tau_{n+1}^m (\bar{\mathbb{B}}_{12} \tau_n^m) = \frac{\bar{d}'_2}{\bar{d}_2} (\bar{\mathbb{B}}_1 \tau_n^m) (\bar{\mathbb{B}}_2 \tau_{n+1}^m) - \frac{\bar{d}'_1}{\bar{d}_1} (\bar{\mathbb{B}}_1 \tau_{n+1}^m) (\bar{\mathbb{B}}_2 \tau_n^m) \tag{5.14}$$

$$\bar{\beta}_{12} \tau_{n+1}^{m+1} (\bar{\mathbb{B}}_{12} \tau_n^{m-1}) = \frac{\bar{a}_1 \bar{a}'_2}{\bar{d}_1 \bar{d}_2} (\bar{\mathbb{B}}_1 \tau_n^m) (\bar{\mathbb{B}}_2 \tau_{n+1}^m) - \frac{\bar{a}'_1 \bar{a}_2}{\bar{d}_1 \bar{d}_2} (\bar{\mathbb{B}}_1 \tau_{n+1}^m) (\bar{\mathbb{B}}_2 \tau_n^m) \tag{5.15}$$

and the $\mathbb{B}\text{-}\bar{\mathbb{B}}$ products

$$\gamma_{12} \tau_n^m (\mathbb{B}_1 \bar{\mathbb{B}}_2 \tau_n^m) = (\mathbb{B}_1 \tau_n^m) (\bar{\mathbb{B}}_2 \tau_n^m) - \frac{a_1 \bar{d}'_2}{d_1 \bar{a}_2} (\mathbb{B}_1 \tau_{n+1}^m) (\bar{\mathbb{B}}_2 \tau_n^{m-1}) \tag{5.16}$$

$$\gamma_{12} \tau_{n+1}^{m-1} (\mathbb{B}_1 \bar{\mathbb{B}}_2 \tau_n^m) = \frac{d'_1 \bar{d}_2}{d_1 \bar{a}_2} (\mathbb{B}_1 \tau_{n+1}^m) (\bar{\mathbb{B}}_2 \tau_n^{m-1}) + \frac{\bar{d}'_2}{\bar{a}_2} (\mathbb{B}_1 \tau_n^m) (\bar{\mathbb{B}}_2 \tau_{n+1}^{m-1}) \tag{5.17}$$

$$\gamma_{12} \tau_{n+1}^{m+1} (\mathbb{B}_1 \bar{\mathbb{B}}_2 \tau_n^m) = \frac{a_1 \bar{a}'_2}{d_1 \bar{a}_2} (\mathbb{B}_1 \tau_{n+1}^m) (\bar{\mathbb{B}}_2 \tau_{n+1}^m) + \frac{a'_1}{d_1} (\mathbb{B}_1 \tau_{n+1}^m) (\bar{\mathbb{B}}_2 \tau_n^m). \tag{5.18}$$

Here, again, $\bar{\beta}_{12}$ and γ_{12} are some constants which depend on the boundary conditions and their explicit form should be established after specifying the class of solutions we are dealing with.

After we have obtained explicit formulae for the superposition of two BTs, it is easy to generalize them to an arbitrary number of BTs involved by noting their determinant structure. For example, equation (5.11) can be rewritten as

$$\mathbb{B}_{12} \tau_n^m = \frac{1}{\beta_{12}} \frac{1}{\tau_{n+1}^m} \det \left| \begin{pmatrix} (\mathbb{B}_1 \tau_n^m) & (\mathbb{B}_2 \tau_n^m) \\ \frac{a'_1}{a_1} (\mathbb{B}_1 \tau_{n+1}^m) & \frac{a'_2}{a_2} (\mathbb{B}_2 \tau_{n+1}^m) \end{pmatrix} \right| \tag{5.19}$$

and generalized as

$$\tau_n^m [N] = \frac{\Gamma[N]}{\prod_{j=1}^{N-1} \tau_{n+j}^m} \det \left| \left(\frac{a'_k}{a_k} \right)^{j-1} (\mathbb{B}_k \tau_{n+j-1}^m) \right|_{j,k=1,\dots,N} \tag{5.20}$$

where the traditional designation $\tau_n^m[N]$ is used for the N times transformed tau-function,

$$\tau_n^m[N] = \mathbb{B}_1 \cdots \mathbb{B}_N \tau_n^m, \tag{5.21}$$

and the constant $\Gamma[N]$ is given by

$$\Gamma[N] = \left[\prod_{1 \leq j < k \leq N} \beta_{jk} \right]^{-1}. \tag{5.22}$$

A proof of this formula can be given as follows. Subtracting from the j th column, $j = 1, \dots, N - 1$, the $(j + 1)$ th one multiplied by the factor $(a_N/a'_N)(\mathbb{B}_N \tau_{n+j-1}^m / \mathbb{B}_N \tau_{n+j}^m)$ one can make all elements of the N th row equal to zero, except the element at the (N, N) th place. In such a way the N th order determinant at the right-hand side of (5.20) is reduced to the one of order $N - 1$. The elements of the resulting determinant can be shown to be proportional to $\mathbb{B}_{Nk} \tau_{n+j-1}^m$. So, the larger determinant can be presented as the result of the application of \mathbb{B}_N to the smaller one. Repeating this step one comes to (5.20).

In a similar way, one can obtain other formulae describing the superposition of \mathbb{B} -transformations, $\overline{\mathbb{B}}$ -transformations as well as the ‘mixed’ ones for the superposition of both \mathbb{B} - and $\overline{\mathbb{B}}$ -transformations.

The superposition formula for the joint action of both \mathbb{B} - and $\overline{\mathbb{B}}$ -transformations can be written as

$$\tau_n^m[N, \overline{N}] = \frac{\Gamma[N, \overline{N}]}{\tau_n^\mu \left(\prod_{j=1}^{M-1} \tau_{n+j}^\mu \right) \left(\prod_{\bar{j}=1}^{\overline{M}-1} \tau_{n+\bar{j}}^\mu \right)} \det \mathcal{M}_n^\mu. \tag{5.23}$$

Here,

$$\tau_n^m[N, \overline{N}] = \mathbb{B}_1 \cdots \mathbb{B}_N \overline{\mathbb{B}}_1 \cdots \overline{\mathbb{B}}_{\overline{N}} \tau_n^m, \tag{5.24}$$

the numbers M and \overline{M} , $M + \overline{M} = N + \overline{N}$, are given by

$$M = N - m + \mu, \quad \overline{M} = \overline{N} + m - \mu \tag{5.25}$$

and \mathcal{M}_n^μ is the $(N + \overline{N}) \times (N + \overline{N})$ matrix

$$\mathcal{M}_n^\mu = \begin{pmatrix} \mathcal{A}_n^\mu[M, N] & \mathcal{B}_n^\mu[M, \overline{N}] \\ \mathcal{C}_n^\mu[\overline{M}, N] & \mathcal{D}_n^\mu[\overline{M}, \overline{N}] \end{pmatrix} \tag{5.26}$$

(the numbers in the square brackets determine the sizes of the matrices that form the matrix \mathcal{M}_n^μ : $\mathcal{A}_n^\mu[M, N]$ is a $M \times N$ matrix, etc). The elements of the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} are given by

$$\mathcal{A}_n^\mu[M, N]_{j,\kappa} = \left(\frac{a'_\kappa}{a_\kappa} \right)^{j-1} \mathbb{B}_\kappa \tau_{n+j-1}^\mu \tag{5.27}$$

$$\mathcal{B}_n^\mu[M, \overline{N}]_{j,\bar{\kappa}} = -\frac{\bar{d}_{\bar{\kappa}}}{\bar{a}_{\bar{\kappa}}} \left(-\frac{\bar{a}'_{\bar{\kappa}}}{\bar{a}_{\bar{\kappa}}} \right)^{j-1} \overline{\mathbb{B}}_{\bar{\kappa}} \tau_{n+j-1}^{\mu-1} \tag{5.28}$$

$$\mathcal{C}_n^\mu[\overline{M}, N]_{\bar{j},\kappa} = -\frac{a_\kappa}{d_\kappa} \left(\frac{d'_\kappa}{d_\kappa} \right)^{\bar{j}-1} \mathbb{B}_\kappa \tau_{n+\bar{j}-1}^{\mu+1} \tag{5.29}$$

$$\mathcal{D}_n^\mu[\overline{M}, \overline{N}]_{\bar{j},\bar{\kappa}} = \left(-\frac{\bar{d}'_{\bar{\kappa}}}{\bar{d}_{\bar{\kappa}}} \right)^{\bar{j}-1} \overline{\mathbb{B}}_{\bar{\kappa}} \tau_{n+\bar{j}-1}^\mu. \tag{5.30}$$

In all above formulae

$$j = 1, \dots, M, \quad \bar{j} = 1, \dots, \overline{M}, \quad \kappa = 1, \dots, N, \quad \bar{\kappa} = 1, \dots, \overline{N} \tag{5.31}$$

and the factor $\Gamma[N, \bar{N}]$ is given by

$$\Gamma[N, \bar{N}] = \frac{(-)^{(m-\mu)(N+\bar{N})}}{\beta[N]\gamma[N, \bar{N}]\bar{\beta}[\bar{N}]} \left[\left(\prod_{\kappa=1}^N \frac{d_{\kappa}}{a_{\kappa}} \right) \left(\prod_{\bar{\kappa}=1}^{\bar{N}} \frac{\bar{d}_{\bar{\kappa}}}{\bar{a}_{\bar{\kappa}}} \right) \right]^{m-\mu}, \tag{5.32}$$

with

$$\beta[N] = \prod_{1 \leq j < \kappa \leq N} \beta_{j\kappa}, \quad \gamma[N, \bar{N}] = \prod_{j=1}^N \prod_{\bar{\kappa}=1}^{\bar{N}} \gamma_{j\bar{\kappa}}, \quad \bar{\beta}[\bar{N}] = \prod_{1 \leq j < \bar{\kappa} \leq \bar{N}} \bar{\beta}_{\bar{\kappa}j}. \tag{5.33}$$

To illustrate this general result let us consider the simplest case when \mathcal{M}_n^{μ} is a 2×2 matrix. All binary superposition formulae presented in this section can be obtained from (5.23). The following table contains the values of the parameters N, \bar{N} and μ which lead to (5.8), (5.11)–(5.18):

N	\bar{N}	μ	Equation
2	0	$m - 1$	(5.8)
		m	(5.11)
		$m - 2$	(5.12)
0	2	$m + 1$	(5.13)
		m	(5.14)
		$m + 2$	(5.15)
1	1	m	(5.16)
		$m - 1$	(5.17)
		$m + 1$	(5.18)

6. Solitons of the ALH

In this section, I will discuss two types of soliton solutions: the bright solitons (which correspond to the case of the zero boundary conditions) and the dark ones (the case of the so-called finite-density boundary conditions). They will be obtained by applying the N -fold ($2N$ -fold) BTs to the vacuum solutions. It should be noted that the formulae of the previous section immediately give us the structure of N -soliton solutions. So, we have only to write the dispersion laws and to solve the question of the commutativity of different BTs.

Contrary to traditional IST-based approach, I will distinguish different types of solitons not by writing the boundary conditions for the functions q_n and r_n , but starting from the corresponding vacuum solutions (0-soliton ones) directly in terms of the tau-functions τ_n^m .

6.1. Bright solitons of the ALH

The vacuum tau-function corresponding to the bright soliton case is given by

$$\tau_n^m = \delta_{m,0} \tag{6.1}$$

or, if to return to the original tau-functions, τ_n, σ_n and ρ_n , by

$$\tau_n = 1, \quad \sigma_n = \rho_n = 0. \tag{6.2}$$

Applying the elementary BTs (4.3)–(4.6), one gets

$$\begin{aligned}
 \mathbb{B}^{(1)}\tau_n &= (d/d')^n & \mathbb{B}^{(1)}\sigma_n &= 0 & \mathbb{B}^{(1)}\rho_n &= 0 \\
 \mathbb{B}^{(2)}\tau_n &= 0 & \mathbb{B}^{(2)}\sigma_n &= (-a/a')^n & \mathbb{B}^{(2)}\rho_n &= 0 \\
 \mathbb{B}^{(3)}\tau_n &= (d/d')^n & \mathbb{B}^{(3)}\sigma_n &= 0 & \mathbb{B}^{(3)}\rho_n &= 0 \\
 \mathbb{B}^{(4)}\tau_n &= 0 & \mathbb{B}^{(4)}\sigma_n &= (-a/a')^n & \mathbb{B}^{(4)}\rho_n &= 0.
 \end{aligned}
 \tag{6.3}$$

It is easy to note that

$$\mathbb{B}^{(1)} = \mathbb{B}^{(3)}, \quad \mathbb{B}^{(2)} = \mathbb{B}^{(4)}
 \tag{6.4}$$

which means that general BT, \mathbb{B} , can be taken as a combination of two elementary one, say, as

$$\mathbb{B} = u^{(2)}\mathbb{B}^{(2)} + u^{(3)}\mathbb{B}^{(3)},
 \tag{6.5}$$

and to ensure the commutativity it is sufficient to impose the restriction

$$a' = d'
 \tag{6.6}$$

and to choose the functions $\lambda(\xi)$ and $\mu(\eta)$ as follows:

$$\lambda(\xi) = 1, \quad \mu(\eta) = 1 - \eta\zeta, \quad \zeta = -\frac{d}{a}
 \tag{6.7}$$

which leads to equations (5.8), (5.11) and (5.12) with

$$\beta_{12} = \zeta_1 - \zeta_2.
 \tag{6.8}$$

Calculating the coefficients that appear in the superposition of the Miwa’s shifts

$$a(\xi, \eta) = \bar{a}(\xi, \eta) = \xi - \eta, \quad e(\xi, \eta) = 1 - \xi\eta
 \tag{6.9}$$

and

$$c(\xi, \eta) = \bar{c}(\xi, \eta) = g(\xi, \eta) = 1,
 \tag{6.10}$$

which is easy to do by the substitution of (6.1) in the formulae presented in the appendix, one comes to the following equations describing the dependence of the functions $u^{(k)}$ on the variables of the ALH:

$$\begin{aligned}
 \frac{\mathbb{E}_\xi u^{(2)}}{u^{(2)}} &= 1 & \frac{\bar{\mathbb{E}}_\eta u^{(2)}}{u^{(2)}} &= \frac{1}{1 - \eta\zeta} \\
 \frac{\mathbb{E}_\xi u^{(3)}}{u^{(3)}} &= 1 - \xi/\zeta & \frac{\bar{\mathbb{E}}_\eta u^{(3)}}{u^{(3)}} &= 1.
 \end{aligned}
 \tag{6.11}$$

Similar analysis can be applied to the $\bar{\mathbb{B}}$ -transformations. After calculating the elementary transformations

$$\begin{aligned}
 \bar{\mathbb{B}}^{(1)}\tau_n &= (\bar{a}/\bar{a}')^n & \bar{\mathbb{B}}^{(1)}\sigma_n &= 0 & \bar{\mathbb{B}}^{(1)}\rho_n &= 0 \\
 \bar{\mathbb{B}}^{(2)}\tau_n &= 0 & \bar{\mathbb{B}}^{(2)}\sigma_n &= 0 & \bar{\mathbb{B}}^{(2)}\rho_n &= (-\bar{d}/\bar{d}')^n \\
 \bar{\mathbb{B}}^{(3)}\tau_n &= (\bar{a}/\bar{a}')^n & \bar{\mathbb{B}}^{(3)}\sigma_n &= 0 & \bar{\mathbb{B}}^{(3)}\rho_n &= 0 \\
 \bar{\mathbb{B}}^{(4)}\tau_n &= 0 & \bar{\mathbb{B}}^{(4)}\sigma_n &= 0 & \bar{\mathbb{B}}^{(4)}\rho_n &= (-\bar{d}/\bar{d}')^n
 \end{aligned}
 \tag{6.12}$$

and noting that

$$\bar{\mathbb{B}}^{(1)} = \bar{\mathbb{B}}^{(3)}, \quad \bar{\mathbb{B}}^{(2)} = \bar{\mathbb{B}}^{(4)},
 \tag{6.13}$$

one can choose

$$\bar{\mathbb{B}} = \bar{u}_2\bar{\mathbb{B}}^{(2)} + \bar{u}_3\bar{\mathbb{B}}^{(3)}
 \tag{6.14}$$

which leads to the restriction

$$\bar{a}' = \bar{d}' \tag{6.15}$$

and

$$\bar{\lambda}(\xi) = 1, \quad \bar{\mu}(\eta) = 1 - \eta/\bar{\zeta}, \quad \bar{\zeta} = -\frac{\bar{a}}{\bar{d}}. \tag{6.16}$$

Further, one can get $\bar{\beta}_{12} = \bar{\zeta}_1 - \bar{\zeta}_2$ and the following equations:

$$\begin{aligned} \frac{\mathbb{E}_\xi \bar{u}^{(2)}}{\bar{u}^{(2)}} &= 1 - \xi \bar{\zeta} & \frac{\bar{\mathbb{E}}_\eta \bar{u}^{(2)}}{\bar{u}^{(2)}} &= 1 \\ \frac{\mathbb{E}_\xi \bar{u}^{(3)}}{\bar{u}^{(3)}} &= 1 & \frac{\bar{\mathbb{E}}_\eta \bar{u}^{(3)}}{\bar{u}^{(3)}} &= \frac{1}{1 - \eta/\bar{\zeta}}. \end{aligned} \tag{6.17}$$

An important note: both \mathbb{B} - and $\bar{\mathbb{B}}$ -transformations, taken alone, destroy the symmetry between q_n and r_n . Say, $\mathbb{B}q_n = \text{const} \times \zeta^{-n}$, while $\mathbb{B}r_n = 0$, which means that it is impossible to make the transformed functions to satisfy the ‘physical’ involution: $q_n = -r_n^*$ (the star stands for the complex conjugation). The same is valid for the $\bar{\mathbb{B}}$ -transformations. Hence, to obtain ‘physical’ solutions in the bright soliton case we have to use the \mathbb{B} - and $\bar{\mathbb{B}}$ -transformations only in combination, which gives possibility of preserving the involution. Say, to obtain the N -soliton solution we have to use the superposition of N \mathbb{B} - and N $\bar{\mathbb{B}}$ -transformations (with imposing some relations on parameters of the former and the latter). Setting

$$\bar{N} = N \tag{6.18}$$

and noting that

$$\mathbb{B}_\kappa \tau_n = \left(\frac{d_\kappa}{d'_\kappa}\right)^n \mathbb{B}_\kappa \tau_0 \quad \mathbb{B}_\kappa \sigma_n = \left(-\frac{a_\kappa}{a'_\kappa}\right)^n \mathbb{B}_\kappa \sigma_0 \tag{6.19}$$

and

$$\bar{\mathbb{B}}_{\bar{\kappa}} \tau_n = \left(\frac{\bar{a}_{\bar{\kappa}}}{\bar{a}'_{\bar{\kappa}}}\right)^n \bar{\mathbb{B}}_{\bar{\kappa}} \tau_0 \quad \bar{\mathbb{B}}_{\bar{\kappa}} \rho_n = \left(-\frac{\bar{d}_{\bar{\kappa}}}{\bar{d}'_{\bar{\kappa}}}\right)^n \bar{\mathbb{B}}_{\bar{\kappa}} \rho_0, \tag{6.20}$$

one can rewrite expression (5.23) for the superposition of BTs, in the $\mu = 0$ case, as

$$\tau_n^m[N, N] = (-)^{\frac{1}{2}m(m+1)} C_* \zeta_*^m \xi_*^n \det \begin{vmatrix} \zeta_\kappa^{J-1} (\mathbb{B}_\kappa \tau_0) & \bar{\zeta}_{\bar{\kappa}}^{-n-J} (\bar{\mathbb{B}}_{\bar{\kappa}} \rho_0) \\ \zeta_\kappa^{-n-J} (\mathbb{B}_\kappa \sigma_0) & \bar{\zeta}_{\bar{\kappa}}^{J-1} (\bar{\mathbb{B}}_{\bar{\kappa}} \tau_0) \end{vmatrix} \begin{matrix} j=1, \dots, N-m \\ j=1, \dots, N+m \\ \kappa, \bar{\kappa}=1, \dots, N \end{matrix} \tag{6.21}$$

where the quantities ζ_* , ξ_* and C_* are given by

$$\zeta_* = (-)^N \frac{\prod_{\kappa=1}^N \zeta_\kappa}{\prod_{\bar{\kappa}=1}^N \bar{\zeta}_{\bar{\kappa}}}, \quad \xi_* = \left(\prod_{\kappa=1}^N \frac{d_\kappa}{d'_\kappa}\right) \left(\prod_{\bar{\kappa}=1}^N \frac{\bar{a}_{\bar{\kappa}}}{\bar{a}'_{\bar{\kappa}}}\right) \tag{6.22}$$

and

$$C_* = \frac{(-)^{\frac{1}{2}N(N-1)}}{\beta[N] \gamma[N, N] \bar{\beta}[N]}. \tag{6.23}$$

That leads to the following formula for the N -soliton solution:

$$q_n[N, N] = -\zeta_* \frac{\Delta_n^{(1)}}{\Delta_n^{(0)}}, \quad r_n[N, N] = \zeta_*^{-1} \frac{\Delta_n^{(-1)}}{\Delta_n^{(0)}} \tag{6.24}$$

where

$$\Delta_n^{(m)} = \Delta_n^{(m)}(z, \bar{z}) = \det \begin{vmatrix} \zeta_\kappa^{j-1} & R_{\bar{\kappa}}(z, \bar{z}) \bar{\zeta}_{\bar{\kappa}}^{-n-j} \\ Q_\kappa(z, \bar{z}) \zeta_\kappa^{-n-j} & \bar{\zeta}_{\bar{\kappa}}^{j-1} \end{vmatrix} \begin{matrix} j=1, \dots, N-m \\ j=1, \dots, N+m \\ \kappa, \bar{\kappa}=1, \dots, N \end{matrix}. \quad (6.25)$$

The quantities $Q_\kappa(z, \bar{z})$ and $R_{\bar{\kappa}}(z, \bar{z})$, which describe the dependence on the ALH coordinates z_j and \bar{z}_j ,

$$Q_\kappa(z, \bar{z}) = \frac{u_\kappa^{(2)}}{u_\kappa^{(3)}}, \quad R_{\bar{\kappa}}(z, \bar{z}) = \frac{\bar{u}_{\bar{\kappa}}^{(2)}}{\bar{u}_{\bar{\kappa}}^{(3)}}, \quad (6.26)$$

can be easily calculated from (6.11) and (6.17) using the identity

$$\ln(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j} \quad (6.27)$$

and are given by

$$Q_\kappa(z, \bar{z}) = Q_\kappa^{(0)} \exp \left\{ -i \sum_{j=1}^{\infty} (z_j \zeta_\kappa^{-j} + \bar{z}_j \zeta_\kappa^j) \right\} \quad (6.28)$$

$$R_{\bar{\kappa}}(z, \bar{z}) = R_{\bar{\kappa}}^{(0)} \exp \left\{ i \sum_{j=1}^{\infty} (z_j \bar{\zeta}_{\bar{\kappa}}^j + \bar{z}_j \bar{\zeta}_{\bar{\kappa}}^{-j}) \right\} \quad (6.29)$$

where $Q_\kappa^{(0)}$ and $R_{\bar{\kappa}}^{(0)}$ are some arbitrary constants (these expressions are nothing but the well-known dispersion laws for the bright solitons).

6.2. Dark solitons of the ALH

The vacuum tau-function corresponding to the so-called finite-density boundary conditions (the dark soliton case) is given by

$$\tau_n^m = \alpha^{\frac{m^2}{2}} \beta^{\frac{n^2}{2}} u^m v^n \quad (6.30)$$

where the constants α and β are related by $\alpha + \beta = 1$, while u and v are some functions of the ALH coordinates. Contrary to the bright solitons which are two parametrical, a dark soliton depends only on one parameter. From the viewpoint of the BTs, this difference manifests itself in the following facts: (1) to construct N -soliton solutions we need to apply N BTs to the vacuum solution (and not $2N$, as was in the case of bright solitons) and (2) the \mathbb{B} - and $\bar{\mathbb{B}}$ -transformations lead to similar results. So, in what follows the N -dark soliton solutions will be obtained as a superposition of the \mathbb{B} -transformations using (5.20). Thus, we have to solve now the following three problems: we have to ensure the commutativity of our BTs, to calculate the dispersion laws and to satisfy the ‘physical’ involution condition $r_n = q_n^*$. As in the previous section, the first problem can be solved rather quickly. The case is that of four elementary BTs (4.3)–(4.6) only two, say $\mathbb{B}^{(1)}$ and $\mathbb{B}^{(3)}$, are independent while $\mathbb{B}^{(2)} = \mathbb{B}^{(3)}$ and $\mathbb{B}^{(4)} = \mathbb{B}^{(1)}$. So, one can define the \mathbb{B} -transformation as

$$\mathbb{B} = u^{(1)} \mathbb{B}^{(1)} + u^{(3)} \mathbb{B}^{(3)}. \quad (6.31)$$

Then it is easy to show that they will commute, if we set

$$d = d' \quad (6.32)$$

and choose

$$\lambda(\xi) = 1 - \xi/\zeta, \quad \mu(\eta) = 1 - \zeta\eta. \tag{6.33}$$

This leads to the superposition formula (5.8) with $\beta_{12} = \zeta_1 - \zeta_2$.

The dependence of $u^{(1,3)}$ on the ALH times is given by

$$\begin{aligned} \frac{\mathbb{E}_\xi u^{(1)}}{u^{(1)}} &= \frac{1}{c(\zeta, \xi)} & \frac{\overline{\mathbb{E}}_\eta u^{(1)}}{u^{(1)}} &= \frac{1}{g(\zeta, \eta)} \\ \frac{\mathbb{E}_\xi u^{(3)}}{u^{(3)}} &= \frac{1}{g(\xi, 1/\zeta)} & \frac{\overline{\mathbb{E}}_\eta u^{(3)}}{u^{(3)}} &= \frac{1}{\bar{c}(1/\zeta, \eta)} \end{aligned} \tag{6.34}$$

where c, \bar{c} and g are the coefficients which appear in formulae describing the superposition of the Miwa’s shifts \mathbb{E} and $\overline{\mathbb{E}}$ (see the appendix). Their explicit form, as well as the explicit form of the functions u and v from (6.30), can be obtained by the substitution of the vacuum tau-function (6.30) into the corresponding equation. This gives

$$c(\xi, \eta) = \bar{c}(\xi, \eta) = \alpha \left(\frac{\mathbb{E}_\xi u}{u} \right) \left(\frac{\mathbb{E}_\eta u}{u} \right) + \beta \left(\frac{\mathbb{E}_\xi v}{v} \right) \left(\frac{\mathbb{E}_\eta v}{v} \right) \tag{6.35}$$

$$g(\xi, \eta) = 1 - \alpha\xi\eta \left(\frac{\mathbb{E}_\xi u}{u} \right) \left(\frac{\overline{\mathbb{E}}_\eta u}{u} \right) \tag{6.36}$$

and

$$\begin{aligned} \frac{\mathbb{E}_\zeta u}{u} &= \frac{\overline{\mathbb{E}}_\zeta u}{u} = \frac{2}{R(\zeta) - \zeta + 1} \\ \frac{\mathbb{E}_\zeta v}{v} &= \left(\frac{\overline{\mathbb{E}}_\zeta v}{v} \right)^{-1} = \frac{2}{R(\zeta) + \zeta + 1} \end{aligned} \tag{6.37}$$

where

$$R^2(\zeta) = 1 + 2(\alpha - \beta)\zeta + \zeta^2. \tag{6.38}$$

The last equations contain all necessary to obtain the dispersion laws that we need. However, the presence of radicals $R(\zeta)$ complicates the following analysis, so, I would like to rewrite them using the parameterization

$$\zeta = \frac{\sin 2\theta(\zeta)}{\sin[2\omega - 2\theta(\zeta)]} \tag{6.39}$$

where the constant parameter ω is given by

$$\alpha = \cos^2 \omega, \quad \beta = \sin^2 \omega. \tag{6.40}$$

In terms of $\theta(\zeta)$,

$$R(\zeta) = \frac{\sin 2\omega}{\sin[2\omega - 2\theta(\zeta)]} \tag{6.41}$$

and

$$c(\xi, \eta) = \frac{\cos[\theta(\xi) - \theta(\eta)]}{\cos \theta(\xi) \cos \theta(\eta)} \tag{6.42}$$

$$g(\xi, \eta) = \frac{\sin \omega \sin[\omega - \theta(\xi) - \theta(\eta)]}{\sin[\omega - \theta(\xi)] \sin[\omega - \theta(\eta)]}, \tag{6.43}$$

and we can represent \mathbb{B} as

$$\mathbb{B}\tau_n^m = \tau_n^m \left(\frac{d}{a}\right)^m [w^{(1)}B_n^{(1,m)} + w^{(3)}B_n^{(3,m)}] \quad (6.44)$$

with

$$B_n^{(1,m)} = \left[\frac{\cos(\omega - \theta)}{\cos \omega \cos \theta}\right]^m \left[\frac{\sin \omega \sin(\omega - \theta)}{\cos \theta}\right]^n \quad (6.45)$$

$$B_n^{(3,m)} = \left[\frac{\sin(\omega - \theta)}{\cos \omega \sin \theta}\right]^m \left[\frac{\sin \omega \cos(\omega - \theta)}{\sin \theta}\right]^n \quad (6.46)$$

(the functions $w^{(1)}$ and $w^{(3)}$ are, up to some unimportant constants, $vu^{(1)}$ and $u^{(3)}$).

From these formulae, one can deduce that the condition $q_n = r_n^*$ can be satisfied by choosing $d/a = \cos \omega$ and introducing the *real* parameters γ_k , describing the k th BT, by $\theta_k = \omega/2 + i\gamma_k$, or

$$\zeta_k = \frac{\sin(\omega + 2i\gamma_k)}{\sin(\omega - 2i\gamma_k)} \quad (6.47)$$

which reproduces the already known fact that the eigenvalues of the scattering problem, corresponding to the dark solitons, are located on the arc of the unit circle $|\zeta_k| = 1$, $|\arg \zeta_k| < \pi - 2\omega$ (in our designations). Now the BTs \mathbb{B}_k can be written as

$$\frac{\mathbb{B}_k \tau_n^m}{\tau_n^m} = b_k \exp[i\vartheta_k m + \chi_k n] \{w_k e^{i\phi_k m} h_k^n + w_k^{-1} e^{-i\phi_k m} h_k^{-n}\}. \quad (6.48)$$

Here, the constant b_k is given by $b_k = \sqrt{vu_k^{(1)}u_k^{(3)} \sin(\frac{\omega}{2} - i\gamma_k) / \cos(\frac{\omega}{2} + i\gamma_k)}$ (I write this expression only for the sake of completeness, because b_k does not appear in the final formulae) while other parameters (more essential) are given by

$$\vartheta_k = \arg \sin(\omega - 2i\gamma_k), \quad \chi_k = \ln \sin \omega + i\vartheta_k \quad (6.49)$$

and

$$h_k = \left| \tan\left(\frac{\omega}{2} + i\gamma_k\right) \right|, \quad \phi_k = \arg \tan\left(\frac{\omega}{2} + i\gamma_k\right). \quad (6.50)$$

The dependence on the ALH coordinates z_j and \bar{z}_j is 'hidden' in the functions $w_k = w_k(z, \bar{z})$ (which are related to the old ones, $u_k^{(1,3)}$, by $w_k \propto \sqrt{vu_k^{(1)}/u_k^{(3)}}$). This dependence is given by equations (6.34) which lead to

$$w_k = w_k^{(0)} \exp\left\{ \sum_{a=1}^{\infty} (\mu_{ka} z_a + \mu_{ka}^* \bar{z}_a) \right\} \quad (6.51)$$

where $w_k^{(0)}$ are the arbitrary constants and μ_{ka}^* are the complex conjugate to μ_{ka} while the last ones are determined by

$$\exp\left\{ 2i \sum_{a=1}^{\infty} \mu_{ka} \frac{\zeta^a}{a} \right\} = \frac{\tan\left[\frac{\omega}{2} + i\gamma_k - \theta(\zeta)\right]}{\tan\left[\frac{\omega}{2} + i\gamma_k\right]}. \quad (6.52)$$

Applying $\zeta \frac{d}{d\zeta}$ to the logarithm of the last equation, one can conclude that the quantities μ_{ka} are the coefficients of the following Taylor series:

$$\sum_{a=1}^{\infty} \mu_{ka} \zeta^a = \frac{i \sin 2\omega}{2 \sin 2\theta_k} \frac{\zeta}{R(\zeta)} \frac{1}{1 - \zeta \exp(2i\vartheta_k)}. \quad (6.53)$$

Now we have all necessary to write the final formula for the N -dark soliton solution of the ALH:

$$q_n[N] = q_{\text{vac}} \exp\left(i \sum_{\kappa} \vartheta_{\kappa}\right) \frac{\Delta_n^{(+)}}{\Delta_n^{(0)}} \quad r_n[N] = q_n^*[N] \tag{6.54}$$

with

$$\Delta_n^{(0)} = \det \left\{ \left\{ w_{\kappa} h_{\kappa}^{n+j-1} + w_{\kappa}^{-1} h_{\kappa}^{-n-j+1} \right\} \exp(-i\vartheta_{\kappa} j) \right\}_{j,k=1,\dots,N} \tag{6.55}$$

$$\Delta_n^{(+)} = \det \left\{ \left\{ e^{i\phi_{\kappa}} w_{\kappa} h_{\kappa}^{n+j-1} + e^{-i\phi_{\kappa}} w_{\kappa}^{-1} h_{\kappa}^{-n-j+1} \right\} \exp(-i\vartheta_{\kappa} j) \right\}_{j,k=1,\dots,N}. \tag{6.56}$$

Here, $q_{\text{vac}} = \sqrt{\alpha}u$ is given by

$$q_{\text{vac}}(z, \bar{z}) = \sqrt{\alpha} \exp \left\{ i \sum_{a=1}^{\infty} \lambda_a (z_a + \bar{z}_a) \right\} \tag{6.57}$$

where λ_a are the coefficients of the Taylor series

$$\sum_{a=1}^{\infty} \lambda_a \zeta^a = \frac{R(\zeta) - \zeta - 1}{2R(\zeta)} \tag{6.58}$$

(which follows from (6.37)).

7. Conclusion

The main idea of this work was to analyse the equations describing the BTs for the ALH, using the so-called functional representation, when an infinite number of the evolutionary equations are replaced using the Miwa’s shifts with a few difference equations. These equations contain much more information than any of the equations of the hierarchy taken alone and this gave us the possibility of *solving* the BT equations. The main result of this work is *explicit* (though formal) solution of the BT equations, i.e. *explicit* form of the BTs. The fact that this can be done is not surprising and is already known from the general Sato’s theory. For example, in [9] one can find rather comprehensive discussion of how this method works in the context of the KP equation. The solutions of the Bäcklund equations obtained in this paper were constructed of four elementary transformations: $n \rightarrow n \pm 1, m \rightarrow m \pm 1$, Miwa’s shifts and multiplication by the factor of the form $a^m b^n$. Each of these transformations is almost trivial. For example, if we apply them to the dark soliton vacuum (6.30), they coincide and do not change the quantities q_n and r_n . In the case of the bright solitons, the situation is even worse. Say, the transformation $m \rightarrow m \pm 1$ (which is one of the ‘elementary BTs’ of [14]) to the vacuum tau-function (6.1) we will obtain that new tau-functions are given by $\tilde{\sigma}_n = 1$ and $\tilde{\tau}_n = \tilde{\rho}_n = 0$ which means that both \tilde{q}_n and \tilde{r}_n are undefined: $\tilde{q}_n = 1/0, \tilde{r}_n = 0/0$. The key ingredient of the BTs is the linear superposition of different $\mathbb{B}^{(k)}$ and $\bar{\mathbb{B}}^{(k)}$, the superposition which is natural for the approach based on the IST technique [4]. Taking as an example the results of section 6, one could note that the ‘soliton’ exponents (6.28) and (6.29) appear when we solve equations determining the coefficients $u^{(k)}$ and $\bar{u}^{(k)}$ of the linear combinations (6.5) and (6.14).

Another important question is the following one: are the BTs of this paper the standard Darboux–Bäcklund transformations or something different? As it was said in the introduction, in the modern theory of the BTs, based on the IST approach, the BTs appear as the results of the Darboux transformations applied to the auxiliary linear problems. In this paper, we have used another approach (trying to minimize the use of the auxiliary—intermediate, in some

sense—problem) and after having rederived the Bäcklund equations did not return to (2.1). Nevertheless, I would like to state here, without presenting explicit formulae describing the action of our BTs at the linear level, that BT discussed in the present work are indeed the well-known Darboux–Bäcklund transformations of the soliton theory. As an illustration to this fact, one can consider the results of section 6 where the BTs were acting as soliton-adding transformations, which is the most typical example of the Darboux–Bäcklund scheme.

Finally, it should be noted that the approach of the present work (to use functional representation instead of the auxiliary linear problem) is not as sensitive to the boundary conditions as one based on the IST and can be more easily modified to the case of other classes of solutions of the ALH (not only soliton ones but, e.g., quasiperiodic, Wronskian and other) without necessity to elaborate from scratch the corresponding inverse scattering scheme.

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Appendix. Superposition of Miwa's shifts

Here, I would like to present formulae describing the superposition of Miwa's shifts. Some of them were derived in [13], other can be obtained from those of [13] by simple algebra.

The superposition of the positive Miwa's shifts \mathbb{E} can be described by

$$a(\xi, \eta) \tau_n^m (\mathbb{E}_\xi \mathbb{E}_\eta \tau_{n+1}^m) = \xi (\mathbb{E}_\xi \tau_{n+1}^m) (\mathbb{E}_\eta \tau_n^m) - \eta (\mathbb{E}_\xi \tau_n^m) (\mathbb{E}_\eta \tau_{n+1}^m) \quad (\text{A.1})$$

$$a(\xi, \eta) \tau_n^m (\mathbb{E}_\xi \mathbb{E}_\eta \tau_n^{m+1}) = \xi (\mathbb{E}_\xi \tau_n^{m+1}) (\mathbb{E}_\eta \tau_n^m) - \eta (\mathbb{E}_\xi \tau_n^m) (\mathbb{E}_\eta \tau_n^{m+1}) \quad (\text{A.2})$$

$$a(\xi, \eta) \tau_n^m (\mathbb{E}_\xi \mathbb{E}_\eta \tau_{n+1}^{m+1}) = \xi (\mathbb{E}_\xi \tau_{n+1}^{m+1}) (\mathbb{E}_\eta \tau_n^m) - \eta (\mathbb{E}_\xi \tau_n^m) (\mathbb{E}_\eta \tau_{n+1}^{m+1}) \quad (\text{A.3})$$

where $a(\xi, \eta)$ is a constant (with respect to the indices m and n) which should be determined from the boundary conditions. Note that only two of these equations are independent: the third one can be obtained by applying (3.2). In the same manner, one can produce a large number of similar formulae. Another type of superposition formulae for the positive Miwa's shifts can be presented as follows:

$$c(\xi, \eta) (\mathbb{E}_\xi \tau_n^m) (\mathbb{E}_\eta \tau_n^m) = \tau_{n-1}^m (\mathbb{E}_\xi \mathbb{E}_\eta \tau_{n+1}^m) + \tau_n^{m-1} (\mathbb{E}_\xi \mathbb{E}_\eta \tau_n^{m+1}) \quad (\text{A.4})$$

$$c(\xi, \eta) (\mathbb{E}_\xi \tau_n^m) (\mathbb{E}_\eta \tau_{n+1}^m) = \tau_n^m (\mathbb{E}_\xi \mathbb{E}_\eta \tau_{n+1}^m) + \xi \tau_n^{m-1} (\mathbb{E}_\xi \mathbb{E}_\eta \tau_{n+1}^{m+1}) \quad (\text{A.5})$$

$$c(\xi, \eta) (\mathbb{E}_\xi \tau_n^{m+1}) (\mathbb{E}_\eta \tau_n^m) = \tau_n^m (\mathbb{E}_\xi \mathbb{E}_\eta \tau_n^{m+1}) - \eta \tau_{n-1}^m (\mathbb{E}_\xi \mathbb{E}_\eta \tau_{n+1}^{m+1}) \quad (\text{A.6})$$

where $c(\xi, \eta) = (\xi - \eta)/a(\xi, \eta)$.

The superposition of the negative Miwa's shifts $\bar{\mathbb{E}}$ can be described by

$$\bar{a}(\xi, \eta) \tau_{n+1}^m (\bar{\mathbb{E}}_\xi \bar{\mathbb{E}}_\eta \tau_n^m) = \xi (\bar{\mathbb{E}}_\xi \tau_n^m) (\bar{\mathbb{E}}_\eta \tau_{n+1}^m) - \eta (\bar{\mathbb{E}}_\xi \tau_{n+1}^m) (\bar{\mathbb{E}}_\eta \tau_n^m) \quad (\text{A.7})$$

$$\bar{a}(\xi, \eta) \tau_n^m (\bar{\mathbb{E}}_\xi \bar{\mathbb{E}}_\eta \tau_n^{m+1}) = \xi (\bar{\mathbb{E}}_\xi \tau_n^{m+1}) (\bar{\mathbb{E}}_\eta \tau_n^m) - \eta (\bar{\mathbb{E}}_\xi \tau_n^m) (\bar{\mathbb{E}}_\eta \tau_n^{m+1}) \quad (\text{A.8})$$

$$\bar{a}(\xi, \eta)\tau_{n+1}^m(\overline{\mathbb{E}}_\xi\overline{\mathbb{E}}_\eta\tau_n^{m+1}) = \xi(\overline{\mathbb{E}}_\xi\tau_n^{m+1})(\overline{\mathbb{E}}_\eta\tau_{n+1}^m) - \eta(\overline{\mathbb{E}}_\xi\tau_{n+1}^m)(\overline{\mathbb{E}}_\eta\tau_n^{m+1}) \quad (\text{A.9})$$

or by

$$\bar{c}(\xi, \eta)(\overline{\mathbb{E}}_\xi\tau_n^m)(\overline{\mathbb{E}}_\eta\tau_n^m) = \tau_{n+1}^m(\overline{\mathbb{E}}_\xi\overline{\mathbb{E}}_\eta\tau_{n-1}^m) + \tau_n^{m-1}(\overline{\mathbb{E}}_\xi\overline{\mathbb{E}}_\eta\tau_n^{m+1}) \quad (\text{A.10})$$

$$\bar{c}(\xi, \eta)(\overline{\mathbb{E}}_\xi\tau_n^m)(\overline{\mathbb{E}}_\eta\tau_{n+1}^m) = \tau_{n+1}^m(\overline{\mathbb{E}}_\xi\overline{\mathbb{E}}_\eta\tau_n^m) + \eta\tau_{n+1}^{m-1}(\overline{\mathbb{E}}_\xi\overline{\mathbb{E}}_\eta\tau_n^{m+1}) \quad (\text{A.11})$$

$$\bar{c}(\xi, \eta)(\overline{\mathbb{E}}_\xi\tau_n^m)(\overline{\mathbb{E}}_\eta\tau_n^{m+1}) = \tau_n^m(\overline{\mathbb{E}}_\xi\overline{\mathbb{E}}_\eta\tau_n^{m+1}) - \xi\tau_{n+1}^m(\overline{\mathbb{E}}_\xi\overline{\mathbb{E}}_\eta\tau_{n-1}^m) \quad (\text{A.12})$$

where, again, $\bar{c}(\xi, \eta) = (\xi - \eta)/\bar{a}(\xi, \eta)$.

Finally, for the mixed products of \mathbb{E} and $\overline{\mathbb{E}}$ shifts one can get formulae such as

$$e(\xi, \eta)\tau_n^m(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_n^m) = (\mathbb{E}_\xi\tau_n^m)(\overline{\mathbb{E}}_\eta\tau_n^m) - \xi\eta(\mathbb{E}_\xi\tau_{n+1}^m)(\overline{\mathbb{E}}_\eta\tau_{n-1}^m) \quad (\text{A.13})$$

$$e(\xi, \eta)\tau_n^{m-1}(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_n^{m+1}) = (\mathbb{E}_\xi\tau_n^m)(\overline{\mathbb{E}}_\eta\tau_n^m) - (\mathbb{E}_\xi\tau_{n+1}^m)(\overline{\mathbb{E}}_\eta\tau_{n-1}^m) \quad (\text{A.14})$$

and

$$g(\xi, \eta)(\mathbb{E}_\xi\tau_n^m)(\overline{\mathbb{E}}_\eta\tau_n^m) = \tau_n^m(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_n^m) - \xi\eta\tau_n^{m-1}(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_n^{m+1}) \quad (\text{A.15})$$

$$g(\xi, \eta)(\mathbb{E}_\xi\tau_{n+1}^m)(\overline{\mathbb{E}}_\eta\tau_{n-1}^m) = \tau_n^m(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_n^m) - \tau_n^{m-1}(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_n^{m+1}) \quad (\text{A.16})$$

$$g(\xi, \eta)(\mathbb{E}_\xi\tau_{n+1}^{m-1})(\overline{\mathbb{E}}_\eta\tau_n^m) = \tau_{n+1}^{m-1}(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_n^m) - \xi\tau_n^{m-1}(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_{n+1}^m) \quad (\text{A.17})$$

$$g(\xi, \eta)(\mathbb{E}_\xi\tau_{n+1}^{m+1})(\overline{\mathbb{E}}_\eta\tau_n^m) = \tau_n^m(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_{n+1}^{m+1}) - \eta\tau_{n+1}^m(\mathbb{E}_\xi\overline{\mathbb{E}}_\eta\tau_n^{m+1}) \quad (\text{A.18})$$

with $e(\xi, \eta)g(\xi, \eta) = 1 - \xi\eta$.

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